BDDC and FETI-DP under Minimalist Assumptions

Jan Mandel[*‡]   Bedřich Sousedík[*†‡§]

February 1, 2008

Abstract

The FETI-DP, BDDC and P-FETI-DP preconditioners are derived in a particulary simple abstract form. It is shown that their properties can be obtained from only on a very small set of algebraic assumptions. The presentation is purely algebraic and it does not use any particular definition of method components, such as substructures and coarse degrees of freedom. It is then shown that P-FETI-DP and BDDC are in fact the same. The FETI-DP and the BDDC preconditioned operators are of the same algebraic form, and the standard condition number bound carries over to arbitrary abstract operators of this form. The equality of eigenvalues of BDDC and FETI-DP also holds in the minimalist abstract setting. The abstract framework is explained on a standard substructuring example.

1   Introduction

The BDDC and FETI-DP methods are iterative substructuring methods that use coarse degrees of freedom associated with corners, edges or faces between subdomains, and they are currently the most advanced versions of the BDD and FETI families of methods. The BDDC method introduced by Dohrmann [3] is a Neumann-Neumann method of Schwarz type [4]. The BDDC method iterates on the system of primal variables reduced to the interfaces between the subdomains, and it can be understood as further development of the BDD method by Mandel [17]. The FETI-DP method by Farhat et al. [5, 6] is a dual method that iterates on a system for Lagrange multipliers that enforce continuity on the interfaces, and it is a further development of the FETI method by Farhat and Roux [7]. Algebraic relations between FETI and BDD methods were pointed out by Rixen et al. [23], Klawonn and Widlund [12], and Fragakis and Papadrakakis [9]. A common bound on the condition number of both the FETI and the BDD method in terms of a single inequality in was given in [12]. In the case of corner constraints only, a method same as BDDC was suggested by Cros [2]. Fragakis and Papadrakakis [9] derived primal versions of FETI and FETI-DP, called respectively P-FETI and P-FETI-DP, and they have also observed that the eigenvalues of BDD and a certain version of FETI are identical. Mandel, Dohrmann, and Tezaur [19] have proved that the eigenvalues of BDDC and FETI-DP are identical and they have obtained a simplifed and fully algebraic version (i.e., with no undetermined constants) of a common condition number estimate for BDDC and FETI-DP, similar to the estimate by Klawonn and Widlund [12] for BDD and

FETI. Simpler proofs of the equality of eigenvalues of BDDC and FETI-DP were obtained by Li and Widlund [15], and by Brenner and Sung [1], who also gave an example when BDDC has an eigenvalue equal to one but FETI-DP does not. The proof of the equality of eigenvalues of BDD and a certain version of FETI and of FETI-DP and P-FETI-DP was recently given by Fragakis [8] in a more general framework.

In this contribution, we derive the FETI-DP, BDDC and P-FETI-DP preconditioners in a particulary simple abstract form, with only a very small set of algebraic assumptions (Sec. 3). The presentation is purely algebraical and it does not use any particular definition of method components, such as substructures and coarse degrees of freedom. We show that P-FETI-DP and BDDC are in fact the same. We then present the condition number bound and the proof of the equality of eigenvalues of BDDC and FETI-DP, in the minimalist abstract setting (Sec. 4). Finally, we illustrate the abstract framework on a substructuring example (Sec. 5).

2   Notation and Preliminaries

All spaces in this paper are finite dimensional linear spaces. The dual space of a space V is denoted by V' and $\langle \cdot, \cdot \rangle$ is the duality pairing. For a linear operator $L : W \to V$ we define its transpose $L^T : V' \to W'$ by $\langle v, Lw \rangle = \langle L^T v, w \rangle$ for all $v \in V'$, $w \in W$, and $\|v\|_K = \sqrt{\langle Kv, v \rangle}$ denotes the norm associated with a symmetric and positive definite operator $K : V \to V'$, i.e., such that $\langle Kv, v \rangle > 0$ for all $v \in V$, $v \neq 0$. The norm of a linear operator $E : V \to V$ subordinate to this vector norm is defined by $\|E\|_K = \max_{v \in V, v \neq 0} \|Ev\|_K / \|v\|_K$. The notation I_V denotes the identity operator on the space V.

Mappings from a space to its dual arise naturally in the variational setting of systems of linear algebraic equations. An an example, consider an $n \times n$ matrix A and the system of equations $Ax = b$. The variational form of this system is

$$x \in V : (Ax, y) = (b, y) \quad \forall y \in V,$$

where $V = \mathbb{R}^n$ and $(\cdot, \cdot)$ is the usual Euclidean inner product on $\mathbb{R}^n$. For a fixed x, instead of the value Ax, we find it convenient to consider the linear mapping $y \mapsto (Ax, y)$. This mapping is an element of the dual space V'. Denote this mapping by Kx and its value at y by $\langle Kx, y \rangle$; then $K : V \to V'$ is a linear operator from V to its dual that corresponds to A. This setting involving dual spaces is convenient and compact when dealing with multiple nested spaces, or with dual methods (such as FETI). Restricting a linear functional to a subspace is immediate, while the equivalent notation without duality requires introducing new operators, namely projections or transposes of injections. Also, this setting allows us to make a clear distinction between an approximate solution and its residual, which is in the dual space. It is beneficial to have approximate solutions and residuals in different spaces, because they need to be treated differently.

We wish to solve a system of linear algebraic equations

$$Ku = f,$$

where $K : V \to V'$, by a preconditioned conjugate gradient method. Here, a preconditioner is a mapping $M : V' \to V$. In iteration k the method computes the residual

$$r^{(k)} = Ku^{(k)} - f \in V',$$

and the preconditioner computes the increment to the approximate solution $u^{(k)}$ as a linear combination of the preconditioned residual $Mr^{(k)} \in V$ with preconditioned residuals in earlier

iterations. Convergence properties of the method can be established from the eigenvalues λ of the *preconditioned operator MK*; the condition number

$$\kappa = \frac{\lambda_{\max}(MK)}{\lambda_{\min}(MK)},$$

gives a well-known bound on the error reduction, cf. e.g. [10],

$$\left\| e^{(k)} \right\|_K \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k \left\| e^{(0)} \right\|_K,$$

where $e^{(k)} = u^{(k)} - u$ is the error of the solution in iteration k.

3 Abstract Formulation of the Preconditioners

3.1 Setting and Assumptions

We now list a minimalist set of spaces, linear operators, and assumptions needed to formulate the BDDC and the FETI-DP methods and to prove their properties. Let W be a finite dimensional space and let $a(\cdot, \cdot)$ be a symmetric positive semi-definite bilinear form on W. Let $\widehat{W} \subset W$ be a subspace such that a is positive definite on $\widehat{W}$, and $f \in \widehat{W}'$. We wish to solve a variational problem

$$u \in \widehat{W} : a(u, v) = \langle f, v \rangle \quad \forall v \in \widehat{W}. \tag{1}$$

The preconditioners we are interested in are characterized by a selection of an intermediate space $\widetilde{W}$,

$$\widehat{W} \subset \widetilde{W} \subset W, \tag{2}$$

such that

$$a(\cdot, \cdot) \text{ is positive definite on } \widetilde{W}, \tag{3}$$

and a selection of linear operators E, B, and B_D. The operator E is a projection onto $\widehat{W}$,

$$E : \widetilde{W} \to \widehat{W}, \quad E^2 = E, \quad \text{range } E = \widehat{W}. \tag{4}$$

The role of the operator B is to enforce the condition $u \in \widehat{W}$ by $Bu = 0$,

$$B : \widetilde{W} \to \Lambda, \quad \text{null } B = \widehat{W}, \quad \text{range } B = \Lambda. \tag{5}$$

The operator B_D^T is a generalised inverse of B,

$$B_D^T : \Lambda \to \widetilde{W}, \quad BB_D^T = I_\Lambda. \tag{6}$$

The properties (2) – (6) are enough for the BDDC and the FETI-DP theories separately, and they will be assumed from now on. To relate the two methods, we shall also assume that

$$B_D^T B + E = I \tag{7}$$

when needed. No further assumptions are made in the rest of the paper.

To formulate the preconditioners and their properties, we need to define several more linear operators from the concepts already introduced. Denote by

$$R : \widehat{W} \to \widetilde{W}, \quad R : w \in \widehat{W} \longmapsto w \in \widetilde{W}, \tag{8}$$

the natural injection from $\widehat{W}$ to $\widetilde{W}$. Clearly,

$$ER = I_{\widehat{W}}. \tag{9}$$

3

Remark 3.1. *The conditions (3) – (7) are satisfied in the applications of FETI-DP and BDDC methods, see [19] and Sec. 5. Note that the assumption (6) allows the case when B is a matrix that does not have full row rank. All that is needed is to define Λ as range B. In the literature, [19] and references therein, the projection E is often written in the form $E = RD_P R^T$ where R is a mapping of another space (isomorphic to $\widehat{W}$) into W. In the abstract setting here, we choose to formulate the methods in the space $\widehat{W}$ directly, it turns out that the space W is not needed for the theory at all, and R becomes the identity embedding of $\widehat{W}$ into $\widetilde{W}$. The equation (6) is found already in [23, Lemma 1] in a special case. It was extended to form used presently and to cover more general algorithms, and used to obtain important connections between dual and primal substructuring methods in [9, 12].*

We also define the linear operators $\widehat{S}$ and $\widetilde{S}$ associated with the bilinear form a on the spaces $\widehat{W}$ and $\widetilde{W}$, respectively, by

$$\widehat{S} : \widehat{W} \to \widehat{W}', \quad \left\langle \widehat{S}v, w \right\rangle = a(v, w) \quad \forall v, w \in \widehat{W}. \tag{10}$$

$$\widetilde{S} : \widetilde{W} \to \widetilde{W}', \quad \left\langle \widetilde{S}v, w \right\rangle = a(v, w) \quad \forall v, w \in \widetilde{W}, \tag{11}$$

From (10), the variational problem (1) becomes

$$\widehat{S}u = f. \tag{12}$$

Further, it follows from (10), (11), and (8), that

$$\widehat{S} = R^T \widetilde{S} R. \tag{13}$$

3.2 BDDC

The following presentation of BDDC follows [20]. It is essentially same as the approach of [1], and related to the concept of subassembly in [15]. The *BDDC* [3] is the method of preconditioned conjugate gradients applied to the system (12), with the abstract BDDC preconditioner $M_{BDDC} : \widehat{W}' \to \widehat{W}$ defined as

$$M_{BDDC} : r \longmapsto u = Ew, \quad w \in \widetilde{W} : \quad a(w, z) = \langle r, Ez \rangle, \quad \forall z \in \widetilde{W}. \tag{14}$$

For the equivalence of (14) with other formulations of BDDC, see [19, Lemma 7].

From the definitions of $\widetilde{S}$ in (11) and R in (8), it follows that the operator form of the BDDC preconditioner is

$$M_{BDDC} = E\widetilde{S}^{-1}E^T. \tag{15}$$

3.3 FETI-DP

This presentation of FETI-DP follows [19]. The variational problem (1) is equivalent to the minimization

$$\frac{1}{2}a(u, u) - \langle f, u \rangle \to \min \text{ subject to } u \in \widehat{W}. \tag{16}$$

Using

$$\langle f, u \rangle = \langle f, Eu \rangle = \left\langle E^T f, u \right\rangle, \quad u \in \widehat{W},$$

we can write (16) as a constrained minimization problem posed on $\widetilde{W}$,

$$\frac{1}{2}a(u, u) - \left\langle E^T f, u \right\rangle \to \min \text{ subject to } u \in \widetilde{W} \quad \text{and} \quad Bu = 0.$$

Introducing the Lagrangean

$$\mathcal{L}(w,\lambda) = \frac{1}{2}a(u,u) - \langle E^T f, u \rangle + \langle B^T \lambda, u \rangle,$$

where $\lambda \in \Lambda'$ are the Lagrange multipliers, we obtain that problem (16) is equivalent to solving the saddle-point problem [22]

$$\min_{w \in \widetilde{W}} \max_{\lambda \in \Lambda'} \mathcal{L}(w,\lambda). \tag{17}$$

Since

$$\min_{w \in \widetilde{W}} \max_{\lambda \in \Lambda'} \mathcal{L}(w,\lambda) = \max_{\lambda \in \Lambda'} \min_{w \in \widetilde{W}} \mathcal{L}(w,\lambda),$$

it follows that (16) is equivalent to the dual problem

$$\frac{\partial \mathcal{F}(\lambda)}{\partial \lambda} = 0, \tag{18}$$

where

$$\mathcal{F}(\lambda) = \min_{w \in \widetilde{W}} \mathcal{L}(w,\lambda). \tag{19}$$

Problem (18) is equivalent to stationary conditions for the Lagrangean $\mathcal{L}$,

$$\frac{\partial}{\partial w}\mathcal{L}(w,\lambda) \perp \widetilde{W}, \tag{20}$$

$$\frac{\partial}{\partial \lambda}\mathcal{L}(w,\lambda) = 0,$$

which is the same as solving for $w \in \widetilde{W}$ and $\lambda \in \Lambda'$ from the system

$$\begin{array}{rcl} \widetilde{S}w + B^T\lambda &=& E^T f, \\ Bw &=& 0. \end{array} \tag{21}$$

Solving from the first equation in (21) and substituting into the second equation, we get the dual problem in an operator form,

$$B\widetilde{S}^{-1}B^T\lambda = B\widetilde{S}^{-1}E^T f. \tag{22}$$

The *FETI-DP method* [5, 6] is the method of preconditioned conjugate gradients applied to the problem (22), with the preconditioner given by

$$M_{FETI-DP} = B_D\widetilde{S}B_D^T. \tag{23}$$

The FETI-DP method solves for the Lagrange multiplier λ. The corresponding primal solution is found as the minimizer of w in (19), or equivalently, from (20), which is the same as the first equation in (21); hence,

$$w = \widetilde{S}^{-1}\left(E^T f - B^T\lambda\right).$$

If λ is the exact solution of the dual problem (18), then $w \in \widehat{W}$ and so $u = w$ is the desired solution of the primal minimization problem (16). However, for approximate solution λ, in general $w \notin \widehat{W}$, and so the primal solution needs to be projected onto $\widehat{W}$. We use the operator E for this purpose. So, for an arbitrary Lagrange multiplier λ, the corresponding approximate solution of the original problem is

$$u = E\widetilde{S}^{-1}\left(E^T f - B^T\lambda\right). \tag{24}$$

Note that the operator E does not play any role in FETI-DP iterations themselves. It only serves to form the right-hand side of the constrained problem (21), and to recover the primal solution.

3.4 P-FETI-DP

The P-FETI-DP preconditioner [9] is based on the approximate solution from the first step of FETI-DP, starting from $\lambda = 0$, and with the residual r as the right-hand side. The primal solution corresponding to the result of this step is the output of the preconditioner. Thus, from (24) with $\lambda = 0$ and $f = r$, we have

$$M_{P-FETI-DP}r = E\widetilde{S}^{-1}E^T r. \tag{25}$$

Comparing (25) with the BDDC preconditioner in (15), we have immediately:

Theorem 3.1. *The P-FETI-DP and the BDDC preconditioners are the same.*

4 Condition Number Bounds and Eigenvalues

From (13) and (15), the preconditioned operator of the BDDC method is

$$P_{BDDC} = \left(E\widetilde{S}^{-1}E^T\right)\left(R^T\widetilde{S}R\right). \tag{26}$$

From (22) and (23), the preconditioned operator of FETI-DP is

$$P_{FETI-DP} = \left(B_D\widetilde{S}B_D^T\right)\left(B\widetilde{S}^{-1}B^T\right). \tag{27}$$

Clearly, both the BDDC and the FETI-DP preconditioned operators have the same general form

$$\left(LA^{-1}L^T\right)\left(T^TAT\right), \tag{28}$$

where $A = \widetilde{S}$ is symmetric, positive definite, and L and T are some linear operators such that

$$LT = I, \tag{29}$$

because of (6) and (9). This important observation was made in [15] in the equivalent form $P\widetilde{S}^{-1}P\widetilde{S} : \operatorname{range} P \to \operatorname{range} P$, where P is a projection, and in the present form in [1].

4.1 Results for the Abstract Form of the Preconditioned Operators

It is interesting that the fundamental eigenvalue estimate can be proved for arbitrary operators of the form (28) – (29). The following lemma was proved in terms of the BDDC preconditioner in [19, Theorem 25], and the proof carries over. Because the translation between the two settings is time consuming, the proof (with some simplifications but no substantial differences) is included here for completeness. The resulting proof of the condition number bound for FETI-DP in Theorem 4.1 below appears to be new.

Lemma 4.1. *Let V and U be finite dimensional vector spaces and $A : V \to V'$ be an SPD operator. If $L : V \to U$ and $T : U \to V$ are linear operators such that $LT = I$ on U, then all eigenvalues λ of the operator $\left(LA^{-1}L^T\right)\left(T^TAT\right)$ satisfy*

$$1 \le \lambda \le \|TL\|_A^2. \tag{30}$$

Proof. The operator $\left(LA^{-1}L^T\right)\left(T^TAT\right)$ is selfadjoint with respect to the inner product $\left\langle T^TATu, v\right\rangle$. So, it is sufficient to bound $\left\langle\left(LA^{-1}L^T\right)\left(T^TAT\right)u, u\right\rangle$ in terms of $\left\langle\left(T^TAT\right)u, u\right\rangle$.

Let $u \in U$. Then

$$\left(LA^{-1}L^T\right)\left(T^TAT\right)u = Lw, \tag{31}$$

where $w = A^{-1}L^TT^TATu$ satisfies

$$w \in V, \quad \langle Aw, v\rangle = \left\langle T^TATu, Lv\right\rangle \quad \forall v \in V. \tag{32}$$

In particular, from (32) with $v = w$ and (31)

$$\langle Aw, w\rangle = \left\langle T^TATu, Lw\right\rangle = \left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle. \tag{33}$$

and using $LT = I$, (32) with $v = Tu$, Cauchy inequality, the definition of transpose, and (33),

$$
\begin{aligned}
\left\langle T^TATu, u\right\rangle^2 &= \left\langle T^TATu, LTu\right\rangle^2 \\
&= \langle Aw, Tu\rangle^2 \\
&\leq \langle Aw, w\rangle\, \langle ATu, Tu\rangle \\
&= \langle Aw, w\rangle\, \left\langle T^TATu, u\right\rangle \\
&= \left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle\left\langle T^TATu, u\right\rangle.
\end{aligned}
$$

Dividing by $\left\langle T^TATu, u\right\rangle$, we get

$$\left\langle T^TATu, u\right\rangle \leq \left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle \quad \forall u \in U,$$

which gives the left inequality in (30).

To prove the right inequality in (30), let again $u \in U$. Then, from (31), Cauchy inequality in the T^TAT inner product, definition of the A norm, properties of the norm, and (33),

$$
\begin{aligned}
&\left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle^2 \\
&= \left\langle T^TATu, Lw\right\rangle^2 \\
&\leq \left\langle T^TATu, u\right\rangle\left\langle T^TATLw, Lw\right\rangle \\
&= \left\langle T^TATu, u\right\rangle\langle ATLw, TLw\rangle \\
&= \left\langle T^TATu, u\right\rangle\|TLw\|_A^2 \\
&\leq \left\langle T^TATu, u\right\rangle\|TL\|_A^2\, \|w\|_A^2 \\
&\leq \left\langle T^TATu, u\right\rangle\|TL\|_A^2\left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle.
\end{aligned}
$$

Dividing by $\left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle$, we get

$$\left\langle T^TATu, \left(LA^{-1}L^T\right)\left(T^TAT\right)u\right\rangle \leq \|TL\|_A^2\left\langle T^TATu, u\right\rangle \quad \forall u \in U.$$

$$\square$$

The lower bound in Lemma 4.1 was also proved in a different way in [1, Lemma 3.4]. The next abstract lemma is the main tool in the comparison of the eigenvalues of the preconditioned operators of BDDC and FETI-DP.

Lemma 4.2 ([1, Lemmas 3.4 – 3.6]). *Let V and U_i, $i = 1, 2$, be finite dimensional vector spaces and $A : V \to V'$ be an SPD operator. If $L_i : V \to U_i$ and $T_i : U_i \to V$ are linear operators such that*

$$L_i T_i = I \text{ on } U_i, \quad i = 1, 2, \tag{34}$$

$$T_1 L_1 + T_2 L_2 = I \text{ on } V, \tag{35}$$

then all eigenvalues (except equal to one) of the operators $\left(L_1 A^{-1} L_1^T\right)\left(T_1^T A T_1\right)$ and $\left(T_2^T A T_2\right)\left(L_2 A^{-1} L_2^T\right)$ are the same, and their multiplicities are identical.

4.2   Results for FETI-DP and BDDC

Condition number bounds now follow immediately from Lemma 4.1.

Theorem 4.1. *The eigenvalues of the preconditioned operators of FETI-DP and BDDC satisfy $1 \le \lambda \le \omega_{FETI-DP}$ and $1 \le \lambda \le \omega_{BDDC}$, respectively, where*

$$\omega_{BDDC} = \|E\|_{\widetilde{S}}^2, \quad \omega_{FETI-DP} = \|B_D^T B w\|_{\widetilde{S}}^2. \tag{36}$$

In addition, if $\widetilde{W} \ne \widehat{W}$ and (7) holds, then also

$$\omega_{BDDC} = \omega_{FETI-DP}. \tag{37}$$

Proof. The eigenvalue bounds with (36) follows from the form of the preconditioned operators (26) and (27) and Lemma 4.1. The equality (37) follows from the fact that E and $B_D B$ are complementary projections by (7), and the norm of a nontrivial projection depends only on the angle between its range and its nullspace [11]. $\qquad \square$

The result in Theorem 4.1 was proved in a different way in [18] for BDDC and in [21] for FETI-DP. For a simple proof of the bound for BDDC directly from the variational formulation (14), see [20, Theorem 2].

Equality of the eigenvalues of the two methods follows immediately from Lemma 4.2:

Theorem 4.2. *Let (3) – (7) hold. Then, (a) the spectra of the preconditioned operators of BDDC and FETI-DP are the same except possibly for eigenvalue equal to one, and all eigenvalues are larger or equal to one, and (b) the multiplicity of any common eigenvalue different from one is the same, and the multiplicity of the eigenvalue equal to one for FETI-DP is less than or equal to the multiplicity for BDDC.*

Statement (a) of Theorem 4.2 was proved in [19] in a different way, and an elegant simplified proof was given in [15]. Statement (b) was proved in [1]. This presentation uses the fundamental lemma and the approach from [1].

5   Substructuring for a Model Problem

To clarify ideas, we show how the spaces and operators arise in the standard substructuring theory for a model problem obtained by a discretization of the second order scalar elliptic problem. Consider a bounded domain $\Omega \subset \mathbb{R}^d$ decomposed into nonoverlapping subdomains Ω_i, $i = 1, ..., N$, which form a conforming triangulation of the domain Ω. Each subdomain Ω_i, from now called a substructure, is a union of Lagrangean $P1$ or $Q1$ finite elements, and the nodes of the finite
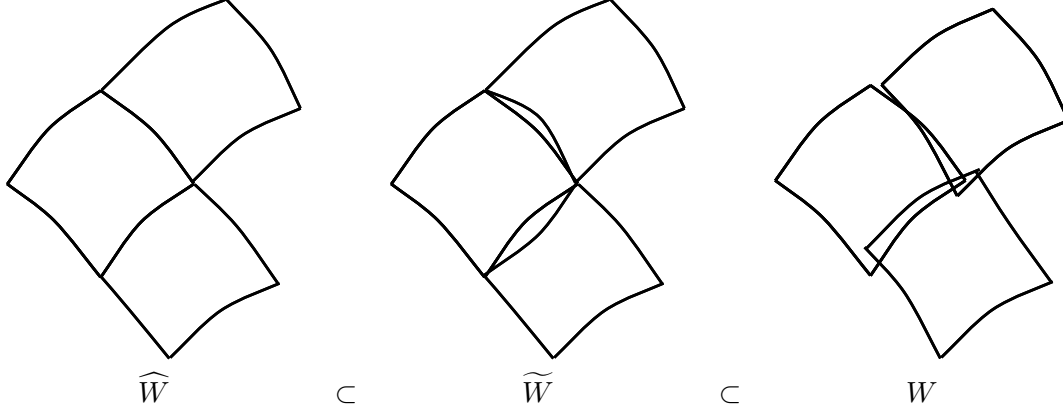
Figure 1: Schematic drawing of continuity conditions between substructures, in the case of corner coarse degrees of freedom only: all degrees of freedom continuous (the space $\widehat{W}$), only the coarse degrees of freedom need to be continuous (the space $\widetilde{W}$), and no continuity conditions (the space W).

elements between the substructures coincide. The nodes contained in the intersection of at least two substructures are called boundary nodes. The union of all boundary nodes of all substructures is called the interface, denoted by Γ, and Γ_i is the interface of substructure Ω_i. The space of all vectors of local degrees of freedom on Γ_i is denoted by W_i. Let $S_i : W_i \to W_i$ be the Schur complement operator obtained from the stiffness matrix of the substructure Ω_i by eliminating all interior degrees of freedom of Ω_i, i.e., those that do not belong to interface Γ_i. We assume that the matrices S_i are symmetric positive semidefinite. Let $W = W_1 \times \cdots \times W_N$ and write vectors and matrices in the block form

$$
w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \quad w \in W, \quad S = \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_N \end{bmatrix}. \tag{38}
$$

The bilinear form a is then given by

$$
a(u, v) = u^T S v.
$$

The solution space $\widehat{W}$ of the problem (1) is a subspace of W such that all subdomain vectors of degrees of freedom are continuous across the interfaces, which here means that their values on all the substructures sharing an interface nodes coincide.

The BDDC and FETI-DP are characterized by selection of *coarse degrees of freedom*, such as values at the corners and averages over edges or faces of subdomains (for their general definition see, e.g., [13]). In the present setting, this becomes the selection of the subspace $\widetilde{W} \subset W$, defined as the subspace of all functions such that coarse degrees of freedom are continuous across the interfaces. Cf., Fig. 1. There needs to be enough coarse degrees of freedom that the variational problem on $\widetilde{W}$ is coercive, i.e., (3) is satisfied. Creating the stiffness matrix on the space $\widetilde{W}$ is called subassembly [16].

The last ingredients are the selections of the linear operators E, B, and B_D. The operators E and B are in fact defined on the whole space W; they are considered restricted on $\widetilde{W}$ only for the purposes of the theory here. The operator $E : W \to \widehat{W}$ is an averaging of the values of degrees of freedom between the substructure. The averaging weights are often taken proportional to the

diagonal entry of the stiffness matrices in the substructures. The matrix B enforces the continuity across substructure interfaces by the condition $Bw = 0$. Each row B has only two nonzero entries, one equal to $+1$ and one equal to -1, corresponding to the two degrees of freedom whose value should be same. So, Bw is the jump of the value of w between substructures. Redundant Lagrange multipliers are possible; then B does not have full row rank and $\Lambda = \text{range } B$ is not the whole euclidean space. Finally, B_D is a matrix such that a vector λ of jumps between the substructures is made into a vector of degrees of freedom $B_D^T \lambda$ that exhibits exactly those jumps. That is, $BB_D^T = I$. The construction of B_D involves weights, related to those in the operator E, so that $B_D^T B + E = I$, and its details are outside of the scope of this paper. Such construction was done first for the FETI method in [14] in order to obtain estimates independent of the jump of coefficients between substructures, and then adopted for FETI-DP. We only note that in many cases of practical relevance, the matrix B_D is determined from the properties (4) – (7) uniquely as the Moore-Penrose pseudoinverse in a special inner product given by the averaging weights in the operator E [19, Theorem 14].

References

[1] S. C. BRENNER AND L.-Y. SUNG, *BDDC and FETI-DP without matrices or vectors*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1429–1435.

[2] J.-M. CROS, *A preconditioner for the Schur complement domain decomposition method*, in Domain Decomposition Methods in Science and Engineering, I. Herrera, D. E. Keyes, and O. B. Widlund, eds., National Autonomous University of Mexico (UNAM), México, 2003, pp. 373–380. 14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico, January 6–12, 2002.

[3] C. R. DOHRMANN, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci. Comput., 25 (2003), pp. 246–258.

[4] M. DRYJA AND O. B. WIDLUND, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math., 48 (1995), pp. 121–155.

[5] C. FARHAT, M. LESOINNE, P. LETALLEC, K. PIERSON, AND D. RIXEN, *FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method*, Internat. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.

[6] C. FARHAT, M. LESOINNE, AND K. PIERSON, *A scalable dual-primal domain decomposition method*, Numer. Linear Algebra Appl., 7 (2000), pp. 687–714. Preconditioning techniques for large sparse matrix problems in industrial applications (Minneapolis, MN, 1999).

[7] C. FARHAT AND F.-X. ROUX, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., 32 (1991), pp. 1205–1227.

[8] Y. FRAGAKIS, *Force and displacement duality in Domain Decomposition Methods for Solid and Structural Mechanics*. To appear in Comput. Methods Appl. Mech. Engrg., 2007.

[9] Y. FRAGAKIS AND M. PAPADRAKAKIS, *The mosaic of high performance domain decomposition methods for structural mechanics: Formulation, interrelation and numerical efficiency of primal and dual methods*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 3799–3830.

[10] G. H. GOLUB AND C. F. V. LOAN, *Matrix Computations*, Johns Hopkins Univ. Press, 1989. Second Edition.

[11] I. C. F. IPSEN AND C. D. MEYER, *The angle between complementary subspaces*, Amer. Math. Monthly, 102 (1995), pp. 904–911.

[12] A. KLAWONN AND O. B. WIDLUND, *FETI and Neumann-Neumann iterative substructuring methods: connections and new results*, Comm. Pure Appl. Math., 54 (2001), pp. 57–90.

[13] A. KLAWONN AND O. B. WIDLUND, *Dual-primal FETI methods for linear elasticity*, Comm. Pure Appl. Math., 59 (2006), pp. 1523–1572.

[14] A. KLAWONN, O. B. WIDLUND, AND M. DRYJA, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., 40 (2002), pp. 159–179.

[15] J. LI AND O. B. WIDLUND, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.

[16] ——, *On the use of inexact subdomain solvers for BDDC algorithms*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1415–1428.

[17] J. MANDEL, *Balancing domain decomposition*, Comm. Numer. Methods Engrg., 9 (1993), pp. 233–241.

[18] J. MANDEL AND C. R. DOHRMANN, *Convergence of a balancing domain decomposition by constraints and energy minimization*, Numer. Linear Algebra Appl., 10 (2003), pp. 639–659. Dedicated to the 70th birthday of Ivo Marek.

[19] J. MANDEL, C. R. DOHRMANN, AND R. TEZAUR, *An algebraic theory for primal and dual substructuring methods by constraints*, Appl. Numer. Math., 54 (2005), pp. 167–193.

[20] J. MANDEL AND B. SOUSEDÍK, *Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1389–1399.

[21] J. MANDEL AND R. TEZAUR, *On the convergence of a dual-primal substructuring method*, Numerische Mathematik, 88 (2001), pp. 543–558.

[22] O. L. MANGASARIAN, *Nonlinear programming*, vol. 10 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994. Corrected reprint of the 1969 original.

[23] D. J. RIXEN, C. FARHAT, R. TEZAUR, AND J. MANDEL, *Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver*, Internat. J. Numer. Methods Engrg., 46 (1999), pp. 501–534.